Knowledge and Decision Systems with Applications

www.kadsa.org
eISSN: 3079-6318

A Comprehensive Study on Fault Detection Techniques for Wheel Bearings in Rotating Machinery: Assessment of the LBF-MABAC model Based on Power Strategies

Muhammad Asif[1], Zeeshan Ali[1,*], Hamza Zafar[1], Yang Tung Chun[1], Afsana Khursheed[1]

[1] Department of Information Management, National Yunlin University of Science and Technology, 123 University Road, Section 3, Douliu, Yunlin 64002, Taiwan, R.O.C.

ARTICLE INFO

Article history:
Received 10 June 2025
Received in revised form 1 December 2025
Accepted 29 December 2025
Available online 2 January 2026

Keywords: Bipolar fuzzy logic; Decision-making analysis; Fault detection; Linguistic term sets; Wheel bearing in rotating machinery.

ABSTRACT

The wheel bearing is a very important part of the car, because they help or supports rotation and reduces friction among moving parts. A detailed investigation and comparison of the many models used to detect faults or failures in wheel bearings, which are critical and complex techniques in rotating machines such as vehicles, turbines, industrial equi#ent, and motors. Four main faults are noticed in the wheel bearing, such as: outer race defects, cage defects, ball/roller defects, and inner race defects, but the most important are preventing catastrophic failures, reducing downtime and repair costs with enables predictive maintenance. The main theme of this study is to choose or develop a technique for engineers' implementation of condition monitoring systems; therefore, first, we design the model of the linguistic bipolar fuzzy technique, then we evaluate the models of "power averaging technique" and "power geometric technique" for linguistic bipolar fuzzy models. Additionally, we also construct the model of the "multi-attribute border approximation area comparison" technique, which is used for the assessment of the fault detection techniques for wheel bearing in rotating machinery. Finally, we illustrate numerical examples to describe the comparative analysis between our ranking values and the ranking values of old models, to mention the advantages and disadvantages of all approaches.

1. Introduction

In a rotatory machine, the wheel bearing plays a fundamental role in the smooth operation. They always reduce the friction and allow all machine parts to rotate efficiently. When bearing faces any fault, then the overall machinery performance is very low. We can identify these types of defects with the help of different machines. Fault-bearing always produces the heat and vibration that cause the power loss. If we ignore these fundamental problems, then we can face many challenges. Fault detection always gives priority and highly focuses on the identification of bearing problems at the

* *Corresponding author.*
E-mail address: zeeshanalinsr@gmail.com;zeeshan@yuntech.edu.tw

https://doi.org/10.59543/kadsa.v2i.14809

early stage with the help of advanced and common techniques. Early identification is always helpful to prevent major damage to any machine components. Early detection reduces the cost and saves time for the repair. Fault detection also enhances the system's reliability and operational safety. In modern days, many industry and smart manufacturing companies apply fault detection to ensure the safety against sudden failures. Many applications have been developed by different scholars because of their validity and effectiveness. For instance, Cao et al. [1] developed a bearing fault in trains based on an empirical wavelet transform. Shaikh et al. [2] investigated the train wheel defect detection. Nawrocki et al. [3] derived the assessment of wear in spindle bearings of machining centers. Yalcin [4] proposed the use of magnetic sensors for the evaluation of the fault detection in tire steel belts. Liu et al. [5] designed the axle box bearing with inner and outer rings. Pandiyan and Babu [6] presented the rolling-element bearing. Hossain et al. [7] evaluated the transform automotive maintenance of bearings. Shaikh et al. [8] determined the fault detection of bearings based on artificial intelligence. Gruber et al. [9] designed the condition monitoring algorithm for the valuation of bearing faults. Feng et al. [10] presented the fault-tolerant collaborative control.

The decision-making process is widely used across multiple domains such as engineering, healthcare, information sciences, supply chain management, and sustainability for solving complex real-world decision-making problems. Most of the real-world decision-making problems contain multiple and conflicting criteria. In this context, the traditional decision-making approaches are limited in evaluating it effectively. To overcome this limitation, Zadeh [11] established the notation of fuzzy set (FS), which is an extension of the existing classical set. FS is an effective mathematical tool that accurately models ambiguity and vagueness in decision-making processes. Unlike the traditional classical set, FS assigns a partial membership function to each element of the set. The FS approach is highly effective for handling uncertain and imprecise expert information. This extension enables the decision-makers to effectively handle real-world decision-making problems. The FS approach is widely extended to different domains. Baser and Ulucay [12] constructed an extended notation using a fuzzy approach and discussed its properties and applications in detail. Kuntama et al. [13] constructed four novel ideas by integrating a fuzzy approach with IUP and explained their properties and applications in detail. Lin et al. [14] constructed a novel conceptual approach of social ecological technological systems based on fuzzy information. Vimala et al. [15] constructed an advanced MADM approach under an extended fuzzy domain for handling a real-world problem of robot selection. Jadhava et al. [16] established a novel solution approach for fuzzy sequential problems under a fuzzy domain. Castello-Sirvent [17] presented a detailed literature review on the articles published in the Journal Citation Report using fuzzy concepts.

Although FS effectively handles complex and uncertain information using a membership function. However, most of the real-world problems contained both positive and negative aspects simultaneously. Such problems cannot be evaluated by using a single membership function. To overcome this limitation, Zhang [18] constructed a novel notation of bipolar fuzzy set (BPS). BPS is an extension of the traditional FS, and it involves two independent membership functions, such as positive membership and negative membership. The positive membership function represents satisfaction, while the negative membership function represents dissatisfaction. The BPS framework enables the expert to express their judgements in an advanced form and can express support and opposition at the same time. The BPS approach is highly effective for situations involving conflicting opinion criteria. Because of it, the BPS gained increasing attention and was widely used across different domains. Gul [19] constructed a novel VIKOR approach under an extended bipolar fuzzy enviro≡ent for handling multi-criteria decision-making (MCDM) problems. Gul et al. [20] established an extended bipolar fuzzified approach under bipolar fuzzy preference relations and discussed its applications in a decision-making enviro≡ent. Dalkılıc and Demirtas [21] constructed a novel decision-making algorithm under an extended bipolar domain for medical diagnosis. Alkouri et al. [22] developed multi attribute decision making approach under an extended bipolar fuzzy domain to find an optimal nutrition program. Ahmad et al. [23] constructed a novel decision-making approach under a generalized bipolar fuzzy enviro≡ent for sustainable energy solutions. Akram and Akmal [24] extended the application of BPS to the graph structural enviro≡ent. Zhang [25] presented a generalized bipolar fuzzy approach and fuzzy equilibrium relations for bipolar clustering, optimization, and global regularization. Akram and Dudek [26] invented a novel notation of regular and total regular graphs under a bipolar fuzzy enviro≡ent and discussed their basic properties in detail. Dalkılıc and Demirtas [27] used an extended bipolar fuzzy approach to handle a real-world problem of medical diagnosis.

The BPS provides an advanced framework to represent complex and uncertain evaluations. However, in many real-world practical applications, experts express their preferences using natural language instead of a fixed numerical value. To model such information, Zadeh [28] invented the concept of linguistic set (LS). The LS framework provides an effective and structured way to model such qualitative expert information. The LS approach reduces information loss while preserving the real meaning of the expert information. This approach enhanced the decision-making approaches and improved interpretability. Because of this, it is widely applied in different enviro≡ents. Gou and Xu [29] constructed some basic operations for LS and extended FS based on linguistic information. Beg and Rashid [30] invented a novel TOPSIS (Technique for Order Preferences by Similarity to Ideal Solution) approach under an extended linguistic enviro≡ent for decision making. Rodriguez et al. [31] constructed a novel group decision-making approach based on generalized linguistic information for managing linguistic expressions. Wang et al. [32] revised the traditional idea of generalized linguistic approach and classified them according to their computational strategies. Sidnyaev et al. [33] invented statistical and linguistic decision-making approaches using fuzzy information.

Based on these ideas, the scholars developed different evaluation approaches to evaluate and rank uncertain and subjective expert information. For example, Pamucar and Cirovic [34] invented a novel MABAC (Multi-Attributive Border Approximation Area Comparison) model for handling problems in logistic centers. He et al. [35] constructed some aggregation operators under an extended linguistic bipolar enviro≡ent and extend its applications to multi-criteria group decision making. Moslem [36] constructed a novel Best-Worst model under extended fuzzy information to investigate commuters' travel mode choices. Moslem [37] constructed an analytic hierarchy process based on generalized fuzzy information for sustainable urban transport solutions. Badi et al. [38] invented a novel MADM model for advanced sustainable logistics and transport systems. Hussain and Ali [39] proposed an advanced decision-making approach under an extended fuzzy domain for critical estimation of ideological and political education. Nwokoro and Ejegwa [40] presented a detailed review on trends, gaps, and future work of an extended fuzzy notation and discussed its application in MCDM. Zafer and Asif [41] developed the N-structure to fuzzy graph, which is a valuable extension of fuzzy set theory. Zhou et al. [42] invented the artificial neural network based on modified fuzzy sets. Jia et al. [43] derived the coefficient bounds for q-calculus.

The fuzzy and its extensions are widely applied across different fields for modeling uncertain and vague expert information. Despite their effectiveness, there are still some major research gaps in the current literature that need to be filled. The major research gaps are as follows:

1) The current literature provides many effective frameworks that enable experts to express their judgements, but the linguistic bipolar fuzzy framework has not developed.
2) The scholars invented many aggregation approaches, but the idea of a power aggregation operator based on a linguistic bipolar fuzzy framework has not constructed.
3) The existing literature contained many MADM approaches, but the notation of the MABAC model based on linguistic bipolar fuzzy data is not established.
4) Many hybrid ideas have been developed, but no one constructed such a framework that integrates power aggregation operators and the MABAC model in one frame.

These major research gaps highly motivate this research work. Our target is to develop such advanced approaches that completely cover the above-mentioned research gaps. We aim to construct the following concepts.

1) Our target is to construct a hybrid notation of linguistic bipolar fuzzy set by using the existing idea of FS, BFS, and LS.
2) Our focus is to extend the existing notation of the power aggregation operators to a linguistic bipolar fuzzy enviro≡ent.
3) We aim to generalize the current idea of the MABAC model and invent a novel MABAC approach under a linguistic bipolar fuzzy enviro≡ent.
4) We also aim to combine the advanced notation of the power aggregation operators with the proposed MABAC model and apply it to a real-world problem.

The linguistic bipolar fuzzy approach is highly effective for handling real-world decision-making problems. It enables the decision makers to model complex and uncertain expert judgements in a reliable way. The key advantages of this study are discussed as follows:

1) The advanced linguistic bipolar fuzzy framework strengthens the decision-making approaches and enables the expert to express their preferences accurately.

2) The modified notation of power aggregation operators effective handle complex and uncertain expert judgements.
3) The generalized MABAC model effectively evaluates and ranks alternatives based on multiple attributes under a complex and uncertain enviro≡ent.
4) The structure of the extended MABAC approach is highly flexible and can easily be extended to different domains.

The advanced linguistic bipolar fuzzy set (LBFS) integrates the strengths of many individual ideas, such as FS, BFS, and LS. The LBFS is a generalization of many ideas, such as FS, BFS, and LS, and all these ideas have now become the special cases of it. Similarly, the invented power aggregation operator and MABAC model is generalization of many existing concepts, such as arithmetic aggregation operators, geometric aggregation operators, power aggregation operators based on fuzzy data, MABAC model, MABAC model based on fuzzy information, and MABAC model based on bipolar fuzzy information, and all these ideas have now become special cases of this research work. In this article, our target is to invent the following ideas.

1) To construct the notation of linguistic bipolar fuzzy set and discuss its basic properties.
2) To develop a generalized idea of power aggregation operators based on linguistic bipolar fuzzy data.
3) To invent an advanced notation of the MABAC model under a linguistic bipolar fuzzy enviro≡ent.
4) To integrate the advanced idea of power aggregation operators with the MABAC model and apply it to a real-world problem.
5) To compare the results of the proposed model with some related current models to check its reliability and accuracy.

This research article is arranged in the following manner. In Section 2, we briefly reviewed the existing idea of BFS and discussed its fundamental properties. In Section 3, we explained the novel notation of linguistic bipolar fuzzy sets and discussed their basic operational laws. Further, we discussed our proposed aggregation operators. Additionally, we explained the modified MABAC model based on linguistic bipolar fuzzy data. In Section 4, we discussed a real-world problem and addressed it with the help of a numerical example. In Section 5, we compared the results of the proposed model with some current related ideas, and in Section 6, we put the concluding remarks.

2. Preliminaries

This section concentrates on the revision of the existing models of the BFS and their operational laws. Further, the idea of linguistic term sets is also discussed with some valuable results.

Definition 1: [18] Let X be a universe of discourse. The BFS W is stated and derived by:

$$W = \{(x, \top\llcorner_{\#W}(x), \top\llcorner_{\equiv W}(x)) \mid x \in X\}$$

Where $\top\llcorner_{\#W}(x) \in [0,1]$ represents the positive value and $\top\llcorner_{\equiv W}(x) \in [-1,0]$ represents the negative value of the truth function. The final version of the BFN is invented and represented by: $W = (\top\llcorner_{\#W}(x), \top\llcorner_{\equiv W}(x))$.

Definition 2: [18] Let $W = (\top\llcorner_{\#W}(x), \top\llcorner_{\equiv W}(x))$ be any BFN. The model of the score system is developed and represented by:

$$\mathfrak{l}_{SF}(W) = \frac{1}{2}\left(1 + \top\llcorner_{\#W}(x) + \top\llcorner_{\equiv W}(x)\right) \in [0,1]$$

Definition 3: [18] Let $W = (\top\llcorner_{\#W}(x), \top\llcorner_{\equiv W}(x))$ be any BFN. The model of the accuracy system is developed and represented by:

$$\mathfrak{A}_{AF}(W) = \frac{1}{2}\left(1 + \top\llcorner_{\#W}(x) - \top\llcorner_{\equiv W}(x)\right) \in [0,1]$$

Definition 4: [18] Let $W_1 = (\top\llcorner_{\#W_1}(x), \top\llcorner_{\equiv W_1}(x))$ and $W_2 = (\top\llcorner_{\#W_2}(x), \top\llcorner_{\equiv W_2}(x))$, be any two BFNs. Then

$$W_1 \oplus W_2 = \left(\top\llcorner_{\#W_1} + \top\llcorner_{\#W_2} - \top\llcorner_{\#W_1}\top\llcorner_{\#W_2}, -\left((\top\llcorner_{\equiv W_1}).(\top\llcorner_{\equiv W_2})\right)\right)$$

$$W_1 \otimes W_2 = \left(\left((\top\llcorner_{\#W_1}).(\top\llcorner_{\#W_2})\right), \top\llcorner_{\equiv W_1} + \top\llcorner_{\equiv W_2} + \top\llcorner_{\equiv W_1}\top\llcorner_{\equiv W_2}\right)$$

$$\lambda W = \left(1 - (1 - \top\llcorner_{\#W})^{\lambda}, -|\top\llcorner_{\equiv W}|^{\lambda}\right)$$

$$W^{\lambda} = \left((\top\llcorner_{\#W})^{\lambda}, -1 + (1 + \top\llcorner_{\equiv W})^{\lambda}\right)$$

Definition 5: [28] Let $\mid = \{\mid_I \mid I = 0,1,\ldots,\surd\}$ be a linguistic term set with specific cardinality. The above sets must simplify the following conditions:

1) If $\mid_I > \mid_j \Leftrightarrow I > j$.
2) $\mathrm{Neg}(\mid_I) = \mid_j$ Such that $j = \surd - I$.
3) $max(\mid_I, \mid_j) = \mid_I \Leftrightarrow I \geq j$.
4) $min(\mid_I, \mid_j) = \mid_j \Leftrightarrow I \leq j$.

3. Assessment of MABAC Model Based on Linguistic Bipolar Fuzzy Power Strategies

This section focuses on the valuation of the MABAC model based on the power aggregation operators for a novel model of linguistic bipolar fuzzy information.

3.1 LBC: Linguistic Bipolar Fuzzy Information

This Section aims to develop the idea of LBFS and its basic and fundamental laws.

Definition 6: Let X be a universe of discourse. The LBFS W is stated and derived by:

$$\mathrm{W} = \left\{\left(x, |_{\top\llcorner_{\sharp \mathrm{W}}(x)}, |_{\top\llcorner_{\equiv \mathrm{W}}(x)}\right) \mid x \in X\right\}$$

Where $\top\llcorner_{\sharp \mathrm{W}}(x) \in [0, \surd], \left(\frac{\top\llcorner_{\sharp \mathrm{W}}(x)}{\surd} \in [0,1]\right)$, represents the positive value and $\top\llcorner_{\equiv \mathrm{W}}(x) \in [-\surd, 0], \left(\frac{\top\llcorner_{\equiv \mathrm{W}}(x)}{\surd} \in [-1,0]\right)$, represents the negative value of the truth function. Thus, $|_{\top\llcorner_{\sharp \mathrm{W}}(x)}$ and $|_{\top\llcorner_{\equiv \mathrm{W}}(x)}$ are called positive and negative linguistic truth values. The final version of the LBFN is invented and represented by: $\mathrm{W} = \left(|_{\top\llcorner_{\sharp \mathrm{W}}(x)}, |_{\top\llcorner_{\equiv \mathrm{W}}(x)}\right)$.

Definition 7: Let $\mathrm{W}_1 = \left(\top\llcorner_{\sharp \mathrm{W}_1}(x), \top\llcorner_{\equiv \mathrm{W}_1}(x)\right)$ and $\mathrm{W}_2 = \left(\top\llcorner_{\sharp \mathrm{W}_2}(x), \top\llcorner_{\equiv \mathrm{W}_2}(x)\right)$ be two LBFNs. Then

$$\mathrm{W}_1 \oplus \mathrm{W}_2 = \left(|_{\surd\left(\frac{\top\llcorner_{\sharp \mathrm{W}_1}}{\surd} + \frac{\top\llcorner_{\sharp \mathrm{W}_2}}{\surd} - \frac{\top\llcorner_{\sharp \mathrm{W}_1}}{\surd}\frac{\top\llcorner_{\sharp \mathrm{W}_2}}{\surd}\right)}, |_{-\surd\left(\frac{(\top\llcorner_{\equiv \mathrm{W}_1})}{\surd} . \frac{(\top\llcorner_{\equiv \mathrm{W}_2})}{\surd}\right)}\right)$$

$$\mathrm{W}_1 \otimes \mathrm{W}_2 = \left(|_{\surd\left(\frac{(\top\llcorner_{\sharp \mathrm{W}_1})}{\surd} . \frac{(\top\llcorner_{\sharp \mathrm{W}_2})}{\surd}\right)}, |_{\surd\left(\frac{\top\llcorner_{\sharp \mathrm{W}_1}}{\surd} + \frac{\top\llcorner_{\sharp \mathrm{W}_2}}{\surd} + \frac{\top\llcorner_{\sharp \mathrm{W}_1}}{\surd}\frac{\top\llcorner_{\sharp \mathrm{W}_2}}{\surd}\right)}\right)$$

$$\lambda \mathrm{W} = \left(|_{\surd\left(1-\left(1-\frac{\top\llcorner_{\sharp \mathrm{W}}}{\surd}\right)^{\lambda}\right)}, |_{-\surd\left(\left(\left|\frac{\top\llcorner_{\equiv \mathrm{W}}}{\surd}\right|\right)^{\lambda}\right)}\right)$$

$$\mathrm{W}^{\lambda} = \left(|_{\surd\left(\frac{\top\llcorner_{\sharp \mathrm{W}}}{\surd}\right)^{\lambda}}, |_{\surd\left(-1+\left(1+\frac{\top\llcorner_{\equiv \mathrm{W}}}{\surd}\right)^{\lambda}\right)}\right)$$

Definition 8: Let $\mathrm{W} = \left(\top\llcorner_{\sharp \mathrm{W}}(x), \top\llcorner_{\equiv \mathrm{W}}(x)\right)$ be any LBFN. Then, the score value is invented and deliberated by:

$$\mathfrak{l}_{\mathrm{SF}}(\mathrm{W}) = \frac{1}{2}\left(\frac{\top\llcorner_{\sharp \mathrm{W}}(x)}{\surd} + \frac{\top\llcorner_{\equiv \mathrm{W}}(x)}{\surd}\right) \in [-1,1]$$

Definition 9: Let $\mathrm{W} = \left(\top\llcorner_{\sharp \mathrm{W}}(x), \top\llcorner_{\equiv \mathrm{W}}(x)\right)$ be any LBFN. Then, the accuracy value is invented and deliberated by:

$$\mathfrak{A}_{\mathrm{AF}}(\mathrm{W}) = \frac{1}{2}\left(\frac{\top\llcorner_{\sharp \mathrm{W}}(x)}{\surd} - \frac{\top\llcorner_{\equiv \mathrm{W}}(x)}{\surd}\right) \in [0,1]$$

3.2 LBF Power Arithmetic Aggregation Operators

In this section, we discuss the arithmetic operators for LBF information and also discuss their properties with the order arithmetic operator.

Definition 10: Let $\mathrm{W}_{\kappa} = \left(|_{\top\llcorner_{\#\mathrm{W}_{\kappa}}(x)}, |_{\top\llcorner_{\equiv\mathrm{W}_{\kappa}}(x)} \right), (\kappa = 1,2,3,\ldots,n)$, be a family of LBFNs. The LBFPWAA operator is invented and deliberated by:

$$\mathrm{LBFPWAA}(\mathrm{W}_1, \mathrm{W}_2, \mathrm{W}_3, \ldots, \mathrm{W}_n) = \oplus_{\kappa=1}^{n}(\Xi_{\kappa}\mathrm{W}_{\kappa}) = \Xi_1\mathrm{W}_1 \oplus \Xi_2\mathrm{W}_2 \oplus \Xi_3\mathrm{W}_3 \oplus \ldots \oplus \Xi_n\mathrm{W}_n$$

Where $\Xi_{\kappa} = \frac{1+\mathbb{H}(\xi_{\kappa})}{\sum_{\kappa=1}^{\mathfrak{P}}(1+\mathbb{H}(\xi_{\kappa}))}$ and $\mathbb{H}(\xi_{\kappa}) = \sum_{\substack{\mu=1 \\ \mu\neq\kappa}}^{\mathfrak{P}} \sup(\partial_{\mathfrak{P}}, \partial_{\propto})$ and $\sup(\partial_{\mathfrak{P}}, \partial_{\propto}) = 1 - d(\partial_{\mathfrak{P}}, \partial_{\propto})$.

Theorem 2: Let $\mathrm{W}_{\kappa} = \left(|_{\top\llcorner_{\#\mathrm{W}_{\kappa}}(x)}, |_{\top\llcorner_{\equiv\mathrm{W}_{\kappa}}(x)} \right), (\kappa = 1,2,3,\ldots,n)$, be a family of LBFNs. Then prove that the aggregated value of the LBFPWAA operator is also an LBFN, such as

$$LBFPWAA(\mathrm{W}_1, \mathrm{W}_2, \mathrm{W}_3, \ldots, \mathrm{W}_n) = \left(|_{\sqrt{}\left(1-\prod_{\kappa=1}^{n}\left(1-\frac{\top\llcorner_{\#\mathrm{W}_{\kappa}}}{\sqrt{}}\right)^{\Xi_{\kappa}}\right)}, |_{-\sqrt{}\left(\prod_{\kappa=1}^{n}\left|\frac{\top\llcorner_{\equiv\mathrm{W}_{\kappa}}}{\sqrt{}}\right|^{\Xi_{n}}\right)} \right)$$

Proof. We will demonstrate this theorem using the principle of mathematical induction. First, we will prove this theorem for $n = 2$, as we know that

$$\Xi_1\mathrm{W}_1 = \left(|_{\boldsymbol{\sigma}\left(\mathbf{1}-\left(1-\frac{\top\llcorner_{\#\mathrm{W}_1}}{\sqrt{}}\right)^{\Xi_1}\right)}, |_{\sqrt{}\left|\frac{\top\llcorner_{\equiv\mathrm{W}_1}}{\sqrt{}}\right|^{\Xi_1}} \right)$$

and

$$\Xi_2\mathrm{W}_2 = \left(|_{\boldsymbol{\sigma}\left(\mathbf{1}-\left(1-\frac{\top\llcorner_{\#\mathrm{W}_2}}{\sqrt{}}\right)^{\Xi_2}\right)}, |_{\sqrt{}\left|\frac{\top\llcorner_{\equiv\mathrm{W}_2}}{\sqrt{}}\right|^{\Xi_2}} \right)$$

Then

$$\Xi_1 W_1 \oplus \Xi_2 W_2 = \left(\left| _{\sigma\left(1-\left(1-\frac{\top\llcorner_{\#W_1}}{\surd}\right)^{\Xi_1}\right)}, \right| _{\surd\left|\frac{\top\llcorner_{\equiv W_1}}{\surd}\right|^{\Xi_1}} \right) \oplus \left(\left| _{\sigma\left(1-\left(1-\frac{\top\llcorner_{\#W_2}}{\surd}\right)^{\Xi_2}\right)}, \right| _{\surd\left|\frac{\top\llcorner_{\equiv W_2}}{\surd}\right|^{\Xi_2}} \right)$$

$$= \left(\left| _{\sigma\left(1-\prod_{\kappa=1}^{2}\left(1-\frac{\top\llcorner_{\#W_\kappa}}{\surd}\right)^{\Xi_\kappa}\right)}, \right| _{-\surd\left(\prod_{\kappa=1}^{2}\left|\top\llcorner_{\equiv W_\kappa}\right|^{\Xi_n}\right)} \right)$$

This implies the given statement is true for $n = 2$. Now, suppose that the given statement is true for $n = k$, such as

$$\mathrm{LBFPWAA}(W_1, W_2, W_3, \ldots, W_k) = \left(\left| _{\surd\left(1-\prod_{\kappa=1}^{k}\left(1-\frac{\top\llcorner_{\#W_\kappa}}{\surd}\right)^{\Xi_\kappa}\right)}, \right| _{-\surd\left(\prod_{\kappa=1}^{k}\left|\frac{\top\llcorner_{\equiv W_\kappa}}{\surd}\right|^{\Xi_\kappa}\right)} \right)$$

Now to prove the given statement is true for $n = k + 1$, such as

$$\mathrm{LBFPWAA}(W_1, W_2, W_3, \ldots, W_k, W_{k+1}) = \oplus_{\kappa=1}^{k}(\Xi_\kappa W_\kappa) \oplus \Xi_{k+1} W_{k+1}$$

$$= \left(\left| _{\surd\left(1-\prod_{\kappa=1}^{k}\left(1-\frac{\top\llcorner_{\#W_\kappa}}{\surd}\right)^{\Xi_\kappa}\right)}, \right| _{-\surd\left(\prod_{\kappa=1}^{k}\left|\top\llcorner_{\equiv W_\kappa}\right|^{\Xi_\kappa}\right)} \right)$$

$$\oplus \left(\left| _{\surd\left(1-\left(1-\frac{\top\llcorner_{\#W_{k+1}}}{\surd}\right)^{\Xi_{k+1}}\right)}, \right| _{-\surd\left(\left|\frac{\top\llcorner_{\equiv W_{k+1}}}{\surd}\right|^{\Xi_{k+1}}\right)} \right)$$

$$= \left(\left| _{\surd\left(1-\prod_{\kappa=1}^{k+1}\left(1-\frac{\top\llcorner_{\#W_\kappa}}{\surd}\right)^{\Xi_\kappa}\right)}, \right| _{-\surd\left(\prod_{\kappa=1}^{k+1}\left|\frac{\top\llcorner_{\equiv W_\kappa}}{\surd}\right|^{\Xi_n}\right)} \right)$$

This implies the given statement is true for $n = k + 1$. This implies the given statement is true for all $n \geq 0$.

Property 1: Let $W_\kappa = \left(\left| _{\top\llcorner_{\#W_\kappa}(x)}, \right| _{\top\llcorner_{\equiv W_\kappa}(x)} \right), (\kappa = 1,2,3, \ldots, n)$, be a family of LBFNs. Then

1) If $W_\kappa = W$ for all κ then $\mathrm{LBFPWAA}(W_1, W_2, W_3, \ldots, W_n) = W$ for all κ.
Proof. Consider $W_\kappa = W$, then,

$$\mathrm{LBFPWAA}(W_1, W_2, W_3, \ldots, W_n) = \left(\left| _{\surd\left(1-\prod_{\kappa=1}^{n}\left(1-\frac{\top\llcorner_{\#W}}{\surd}\right)^{\Xi_\kappa}\right)}, \right| _{-\surd\left(\prod_{\kappa=1}^{n}\left|\frac{\top\llcorner_{\equiv W}}{\surd}\right|^{\Xi_\kappa}\right)} \right)$$

$$= \left(|_{\surd\left(1-\left(1-\frac{\top\llcorner_{\sharp W}}{\surd}\right)^{\sum_{\kappa=1}^{n}\Xi_\kappa}\right)}, |_{-\surd\left(\left|\frac{\top\llcorner_{\equiv W}}{\surd}\right|^{\sum_{\kappa=1}^{n}\Xi_\kappa}\right)} \right)$$

$$= \left(|_{\surd\left(1-\left(1-\frac{\top\llcorner_{\sharp W}}{\surd}\right)\right)}, |_{-\surd\left|\frac{\top\llcorner_{\equiv W}}{\surd}\right|} \right)$$

$$= \left(|_{\surd\left(\frac{\top\llcorner_{\sharp W}}{\surd}\right)}, |_{-\surd\left|\frac{\top\llcorner_{\equiv W}}{\surd}\right|} \right),$$

2) If $\top\llcorner_{\sharp W_\kappa} \leq \top\llcorner_{\sharp W'_\kappa}$, $\top\llcorner_{\equiv W_\kappa} \leq \top\llcorner_{\equiv W'_\kappa}$ then, $\mathrm{LBFPWAA}(W_1, W_2, W_3, \dots, W_n) \leq \mathrm{LBFPWAA}(W'_1, W'_2, W'_3, \dots, W'_n)$.

Proof. Consider that $|_{\top\llcorner_{\sharp W_\kappa}} \leq |_{\top\llcorner_{\sharp W'_\kappa}}$, $|_{\top\llcorner_{\equiv W_\kappa}} \leq |_{\top\llcorner_{\equiv W'_\kappa}}$ for all κ, then,

$$|_{\boldsymbol{\sigma}\left(\mathbf{1}-\frac{\top\llcorner_{\sharp W_\kappa}}{\surd}\right)} \geq |_{\surd\left(1-\frac{\top\llcorner_{\sharp W'_\kappa}}{\surd}\right)}$$

Then,

$$|_{\boldsymbol{\sigma}\left(\prod_{\kappa=1}^{n}\left(1-\frac{\top\llcorner_{\sharp W_\kappa}}{\surd}\right)^{\Xi_\kappa}\right)} \geq |_{\surd\left(\prod_{\kappa=1}^{n}\left(1-\frac{\top\llcorner_{\sharp W'_\kappa}}{\surd}\right)^{\Xi_\kappa}\right)}$$

$$|_{\surd\left(1-\prod_{\kappa=1}^{n}\left(1-\frac{\top\llcorner_{\sharp W_\kappa}}{\surd}\right)^{\Xi_\kappa}\right)} \geq |_{\surd\left(1-\prod_{\kappa=1}^{n}\left(1-\frac{\top\llcorner_{\sharp W'_\kappa}}{\surd}\right)^{\Xi_\kappa}\right)}$$

Similarly,

$$|_{\surd\left(1-\prod_{\kappa=1}^{n}\left(1-\frac{\top\llcorner_{\equiv W_\kappa}}{\surd}\right)^{\Xi_\kappa}\right)} \geq |_{\surd\left(1-\prod_{\kappa=1}^{n}\left(1-\frac{\top\llcorner_{\equiv W'_\kappa}}{\surd}\right)^{\Xi_\kappa}\right)}$$

Further,

$$|_{-\surd\left(\prod_{\kappa=1}^{n}\left|\frac{\top\llcorner_{\sharp W_\kappa}}{\surd}\right|\right)} \geq |_{-\surd\left(\prod_{\kappa=1}^{n}\left|\frac{\top\llcorner_{\sharp W'_\kappa}}{\surd}\right|\right)}$$

$$|_{-\surd\left(\prod_{\kappa=1}^{n}\left|\frac{\top\llcorner_{\equiv W_\kappa}}{\surd}\right|\right)} \geq |_{-\surd\left(\prod_{\kappa=1}^{n}\left|\frac{\top\llcorner_{\equiv W'_\kappa}}{\surd}\right|\right)}$$

Thus, using the score function, we can easily determine our required results, such as

$\mathrm{LBFPWAA}(W_1, W_2, W_3, \dots, W_n) \leq \mathrm{LBFPWAA}(W'_1, W'_2, W'_3, \dots, W'_n)$.

3) If $W^- = \left(\mid_{\underset{\kappa}{min}\{\top\llcorner_{\#W_\kappa}\}}, \mid_{\underset{\kappa}{max}\{\top\llcorner_{\equiv W_\kappa}\}} \right)$ and $W^+ = \left(\mid_{\underset{\kappa}{max}\{\top\llcorner_{\equiv W_\kappa}\}}, \mid_{\underset{\kappa}{min}\{\top\llcorner_{\equiv W_\kappa}\}} \right)$ then,

$$W^- \leq \text{LBFPWAA}(W_1, W_2, W_3, \dots, W_n) \leq W^+.$$

Definition 11: Let $W_\kappa = \left(\mid_{\top\llcorner_{\#W_\kappa}(x)}, \mid_{\top\llcorner_{\equiv W_\kappa}(x)} \right), (\kappa = 1,2,3, \dots, n)$, be a family of LBFNs. The LBFPOWAA operator is invented and deliberated by:

$$\text{LBFPOWAA}(W_1, W_2, W_3, \dots, W_n) = \oplus_{\kappa=1}^{n} \left(\Xi_\kappa W_{o(\kappa)} \right)$$

$$= \Xi_1 W_{o(1)} \oplus \Xi_2 W_{o(2)} \oplus \Xi_3 W_{o(3)} \oplus \dots \oplus \Xi_n W_{o(n)}$$

Where $\left(o(1), o(2), o(3), \dots, o(n)\right)$ is a permutation of $(1,2,3, \dots, n)$ such that $W_{o(\kappa-1)} \geq W_{o(\kappa)}$ for all κ.

Theorem 3: Let $W_\kappa = \left(\mid_{\top\llcorner_{\#W_\kappa}(x)}, \mid_{\top\llcorner_{\equiv W_\kappa}(x)} \right), (\kappa = 1,2,3, \dots, n)$, be a family of LBFNs. Then prove that the aggregated value of the LBFPOWAA operator is also an LBFN, such as

$$\text{LBFPOWAA}(W_1, W_2, W_3, \dots, W_n) = \left(\mid_{\sqrt{\left(1 - \prod_{\kappa=1}^{n} \left(1 - \frac{\top\llcorner_{\#W_{o(\kappa)}}}{\sqrt{}}\right)^{\Xi_\kappa}\right)}}, \mid_{-\sqrt{\left(\prod_{\kappa=1}^{n} \left| \frac{\top\llcorner_{\equiv W_{o(\kappa)}}}{\sqrt{}} \right|^{\Xi_n}\right)}} \right).$$

3.3 LBF Power Geometric Aggregation Operators

In this section, we discuss the geometric operators for LBF information and also discuss their properties with the order geometric operator.

Definition 12: Let $W_\kappa = \left(\mid_{\top\llcorner_{\#W_\kappa}(x)}, \mid_{\top\llcorner_{\equiv W_\kappa}(x)} \right), (\kappa = 1,2,3, \dots, n)$, be a family of LBFNs. The LBFPWGA operator is invented and deliberated by:

$$\text{LBFPWGA}(W_1, W_2, W_3, \dots, W_n) = \otimes_{\kappa=1}^{n} \left((W_\kappa)^{\Xi_\kappa} \right)$$

$$= (W_1)^{\Xi_1} \otimes (W_2)^{\Xi_2} \otimes (W_3)^{\Xi_3} \otimes \dots \otimes (W_n)^{\Xi_n}$$

Theorem 4: Let $W_\kappa = \left(\mid_{\top\llcorner_{\#W_\kappa}(x)}, \mid_{\top\llcorner_{\equiv W_\kappa}(x)} \right), (\kappa = 1,2,3, \dots, n)$, be a family of LBFNs. Then prove that the aggregated value of the LBFPWGA operator is also an LBFN, such as

$$\text{LBFPWGA}(W_1, W_2, W_3, \dots, W_n) = \left(\mid_{\sqrt{\left(\prod_{\kappa=1}^{n} \left(\frac{\top\llcorner_{\#W_\kappa}}{\sqrt{}}\right)^{\Xi_\kappa}\right)}}, \mid_{\sqrt{\left(-1 + \prod_{\kappa=1}^{n} \left(1 + \frac{\top\llcorner_{\equiv W_\kappa}}{\sqrt{}}\right)^{\Xi_\kappa}\right)}} \right)$$

Proof. We will demonstrate this theorem using the principle of mathematical induction. Initially, we will demonstrate the case for $n = 2$, as it serves as the foundational base for our argument.

$$(W_1)^{\Xi_1} = \left(|_{\sqrt{\left(\left(\frac{\top\llcorner \sharp W_1}{\sqrt{}}\right)^{\Xi_1}\right)}}, |_{\sqrt{\left(-1+\left(1+\frac{\top\llcorner \equiv W_1}{\sqrt{}}\right)^{\Xi_1}\right)}} \right)$$

and

$$(W_2)^{\Xi_2} = \left(|_{\sqrt{\left(\left(\frac{\top\llcorner \sharp W_2}{\sqrt{}}\right)^{\Xi_2}\right)}}, |_{\sqrt{\left(-1+\left(1+\frac{\top\llcorner \equiv W_2}{\sqrt{}}\right)^{\Xi_2}\right)}} \right)$$

then

$$(W_1)^{\Xi_1} \otimes (W_2)^{\Xi_2}$$

$$= \left(|_{\sqrt{\left(\frac{\top\llcorner \sharp W_1}{\sqrt{}}\right)^{\Xi_1}}}, |_{\sqrt{\left(-1+\left(1+\frac{\top\llcorner \equiv W_1}{\sqrt{}}\right)\right)}} \right) \otimes \left(|_{\sqrt{\left(\frac{\top\llcorner \sharp W_2}{\sqrt{}}\right)^{\Xi_2}}}, |_{\sqrt{\left(-1+\left(1+\frac{\top\llcorner \equiv W_2}{\sqrt{}}\right)^{\Xi_2}\right)}} \right)$$

$$= \left(|_{\sqrt{\left(\prod_{\kappa=1}^{2}\left(\frac{\top\llcorner \sharp W_\kappa}{\sqrt{}}\right)^{\Xi_\kappa}\right)}}, |_{\sqrt{\left(-1+\prod_{\kappa=1}^{2}\left(1+\frac{\top\llcorner \equiv W_\kappa}{\sqrt{}}\right)^{\Xi_\kappa}\right)}} \right)$$

Now, suppose the given statement is true for $n = k$

$$\text{LBFPWGA}(W_1, W_2, W_3, \dots, W_k) = \left(|_{\sqrt{\left(\prod_{\kappa=1}^{k}\left(\frac{\top\llcorner \sharp W_\kappa}{\sqrt{}}\right)^{\Xi_\kappa}\right)}}, |_{\sqrt{\left(-1+\prod_{\kappa=1}^{k}\left(1+\frac{\top\llcorner \equiv W_\kappa}{\sqrt{}}\right)^{\Xi_\kappa}\right)}} \right)$$

Now, to prove a given statement is true for $n = k + 1$, such as

$$\text{LBFPWGA}(W_1, W_2, W_3, \dots, W_k, W_{k+1}) = \otimes_{\kappa=1}^{k} \left((W_\kappa)^{\Xi_\kappa}\right) \otimes (W_{k+1})^{\Xi_{k+1}}$$

$$= \left(|_{\sqrt{\left(\prod_{\kappa=1}^{k}\left(\frac{\top\llcorner \sharp W_\kappa}{\sqrt{}}\right)^{\Xi_\kappa}\right)}}, |_{\sqrt{\left(-1+\prod_{\kappa=1}^{k}\left(1+\frac{\top\llcorner \equiv W_\kappa}{\sqrt{}}\right)^{\Xi_\kappa}\right)}} \right)$$

$$\otimes \left(|_{\sqrt{\left(\left(\frac{\top\llcorner \sharp W_{k+1}}{\sqrt{}}\right)^{\Xi_{k+1}}\right)}}, |_{\sqrt{\left(-1+\left(1+\frac{\top\llcorner \equiv W_{k+1}}{\sqrt{}}\right)^{\Xi_{k+1}}\right)}} \right)$$

$$= \left(\mid_{\sqrt{\left(\prod_{\kappa=1}^{k+1}\left(\frac{\top\llcorner_{\sharp W_\kappa}}{\sqrt{}}\right)^{\Xi_{k+1}}\right)}}, \mid_{\sqrt{\left(-1+\prod_{\kappa=1}^{k+1}\left(1+\frac{\top\llcorner_{\equiv W_\kappa}}{\sqrt{}}\right)^{\Xi_{k+1}}\right)}} \right)$$

This implies the given equation is true for $n = k + 1$.

Property 2: Let $W_\kappa = \left(\mid_{\top\llcorner_{\sharp W_\kappa}(x)}, \mid_{\top\llcorner_{\equiv W_\kappa}(x)} \right), (\kappa = 1,2,3,\ldots,n)$, be a family of LBFNs. Then

1) If $W_\kappa = W$ for all κ then $LBFPWGA(W_1, W_2, W_3, \ldots, W_n) = W$ for all κ.
2) If $\mid_{\top\llcorner_{\sharp W_\kappa}} \leq \mid_{\top\llcorner_{\sharp W'_\kappa}}$ then, $LBFPWGA(W_1, W_2, W_3, \ldots, W_n) \leq LBFPWGA(W'_1, W'_2, W'_3, \ldots, W'_n)$.
3) If $W^- = \left(\mid_{\underset{\kappa}{min}\{\top\llcorner_{\sharp W_\kappa}\}}, \mid_{\underset{\kappa}{max}\{\top\llcorner_{\equiv W_\kappa}\}} \right)$ and $W^+ = \left(\mid_{\underset{\kappa}{max}\{\top\llcorner_{\equiv W_\kappa}\}}, \mid_{\underset{\kappa}{min}\{\top\llcorner_{\equiv W_\kappa}\}} \right)$ then, $W^- \leq LBFPWGA(W_1, W_2, W_3, \ldots, W_n) \leq W^+$.

Definition 13: Let $W_\kappa = \left(\mid_{\top\llcorner_{\sharp W_\kappa}(x)}, \mid_{\top\llcorner_{\equiv W_\kappa}(x)} \right), (\kappa = 1,2,3,\ldots,n)$, be a family of LBFNs. The LBFPOWGA operator is invented and deliberated by:

$$LBFPOWGA(W_1, W_2, W_3, \ldots, W_n) = \otimes_{\kappa=1}^{n} \left(W_{o(\kappa)}\right)^{\Xi_\kappa}$$

$$= \left(W_{o(1)}\right)^{\Xi_1} \otimes \left(W_{o(2)}\right)^{\Xi_2} \otimes \left(W_{o(3)}\right)^{\Xi_3} \otimes \ldots \otimes \left(W_{o(n)}\right)^{\Xi_n}$$

Where $\left(o(1), o(2), o(3), \ldots, o(n)\right)$ is a permutation of $(1,2,3,\ldots.,n)$ such that $W_{o(\kappa-1)} \geq W_{o(\kappa)}$ for all κ.

Theorem 5: Let $W_\kappa = \left(\mid_{\top\llcorner_{\sharp W_\kappa}(x)}, \mid_{\top\llcorner_{\equiv W_\kappa}(x)} \right), (\kappa = 1,2,3,\ldots,n)$, be a family of LBFNs. Then prove that the aggregated value of the LBFPOWGA operator is also an LBFN, such as

$$LBFPOWGA(W_1, W_2, W_3, \ldots, W_n) = \left(\mid_{\sqrt{\left(\prod_{\kappa=1}^{n}\left(\frac{\top\llcorner_{\sharp W_{o(\kappa)}}}{\sqrt{}}\right)^{\Xi_\kappa}\right)}}, \mid_{\sqrt{\left(-1+\prod_{\kappa=1}^{n}\left(\frac{\top\llcorner_{\equiv W_{o(\kappa)}}}{\sqrt{}}\right)^{\Xi_n}\right)}} \right).$$

3.4 MABAC Model Based on LBF Power Aggregation Information

This section focuses on the valuation of the MABAC model for LBFSs based on power aggregation operators. The MABAC model is an MCDM technique where we find the best solution for the uncertain and complex problems in different fields, like supply chain management, risk management, healthcare, etc. This method starts from the construction of the decision matrix. We arrange all the alternatives and attributes in the form of rows and columns, respectively. Our result totally depends on the construction of this matrix, because all the steps of the MABAC come from this matrix. After arranging all the elements in the decision matrix, we apply the normalization technique to the decision matrix data. For this, we apply the scaler and power multiplication of the power operational laws on the BF entries. Using these laws, we convert all the data into a fixed scale, which is very helpful for finding the best aggregated values. After the calculation of this one, we find the different weights with the help of the power techniques and get the best weight vectors, which play a fundamental role in the aggregation operators. After this one, we apply the LBFPWA and LBFPWG operators on the normalized data and get the steady and flexible aggregated value, which is very fruitful for finding the score value. After the calculations of these steps, we apply the score value formula on the weighted averaging and geometric values and get the overall performance of each alternative. Based on these score values, we give the rank of each alternative and get the best alternatives, which represent the best solution for the fault detection of the wheel bearings. Now we discuss all these steps in the form of a mathematical representation as follows:

Step 1: First, we arrange all the alternatives and attributes of the fault detection techniques in the matrix, which is said to be the decision matrix, and it is the initial step of the MABAC method. The cost and benefits value of this matrix is represented as:

$$\mathcal{B} = \begin{cases} \theta = \langle \mid_{\top\llcorner_{\sharp \mathrm{W}}}, \mid_{\top\llcorner_{\equiv \mathrm{W}}} \rangle & benefits \\ \theta' = \langle \mid_{\surd - \top\llcorner_{\sharp \mathrm{W}}}, \mid_{-\surd - \top\llcorner_{\equiv \mathrm{W}}} \rangle & costs \end{cases}$$

Step 2: With the help of the power and scaler multiplication techniques, we convert all the data into a fixed scale. For this, we apply these two normalization techniques to the decision matrix data and find the best and most accurate normalized values. The mathematical representation of these normalization techniques is:

$$\lambda \mathrm{W} = \left(\mid_{\surd\left(1-\left(1-\frac{\top\llcorner_{\sharp \mathrm{W}}}{\surd}\right)^{\lambda}\right)}, \mid_{\surd\left(-\left|\frac{\top\llcorner_{\equiv \mathrm{W}}}{\surd}\right|^{\lambda}\right)} \right)$$

$$\mathrm{W}^{\lambda} = \left(\mid_{\surd\left(\frac{\top\llcorner_{\sharp \mathrm{W}}}{\surd}\right)^{\lambda}}, \mid_{\surd\left(-1+\left(1+\frac{\top\llcorner_{\equiv \mathrm{W}}}{\surd}\right)^{\lambda}\right)} \right)$$

Step 3: Now apply the LBFPWA and LBFPWG operators on the normalized data and get the flexible and consistent aggregated values. These values play a very important role in the distance calculation. The mathematical representation of these operators is:

$$\mathrm{LBFPWAA}(\mathrm{W}_1, \mathrm{W}_2, \mathrm{W}_3, \dots, \mathrm{W}_n) = \oplus_{\kappa=1}^{n}\left(\Xi_{\kappa} \mathrm{W}_{(\kappa)}\right)$$
$$= \left(\mid_{\surd\left(1-\prod_{\kappa=1}^{n}\left(1-\frac{\top\llcorner_{\sharp \mathrm{W}_{\kappa}}}{\surd}\right)^{\Xi_{\kappa}}\right)}, \mid_{-\surd\left(\prod_{\kappa=1}^{n}\left|\frac{\top\llcorner_{\equiv \mathrm{W}_{\kappa}}}{\surd}\right|^{\Xi_{n}}\right)} \right)$$

and

$$\mathrm{LBFPWGA}(\mathrm{W}_1, \mathrm{W}_2, \mathrm{W}_3, \dots, \mathrm{W}_n) = \otimes_{\kappa=1}^{n}((\mathrm{W}_\kappa)^{\Xi_\kappa}) = \left(\mathrm{I}_{\sqrt{\left(\prod_{\kappa=1}^{n}\left(\frac{\top\llcorner_{\sharp \mathrm{W}_\kappa}}{\sqrt{}}\right)^{\Xi_\kappa}\right)}}, \mathrm{I}_{\sqrt{\left(-1+\prod_{\kappa=1}^{n}\left(1+\frac{\top\llcorner_{\equiv \mathrm{W}_\kappa}}{\sqrt{}}\right)^{\Xi_\kappa}\right)}} \right)$$

Step 4: This is a very important step in this model. In this step, we find the distance between step 2 and step 3. In this way, we calculate how much each alternative is from the ideal boundary. The mathematical representation of this distance value is:

$$d_{\mathcal{s}d} = \begin{cases} d(\mathrm{W}_\mathcal{s}, \mathrm{W}_d), if\ \mathrm{W}_\mathcal{s} > \mathrm{W}_d \\ 0, if\ \mathrm{W}_\mathcal{s} = \mathrm{W}_d \\ -d(\mathrm{W}_\mathcal{s}, \mathrm{W}_d), if\ \mathrm{W}_\mathcal{s} < \mathrm{W}_d \end{cases}$$

$$d(\mathrm{W}_\mathcal{s}, \mathrm{W}_d) = \frac{\mathrm{I}}{2\sqrt{}}\left(\left|\top\llcorner_{\sharp \mathrm{W}_\kappa} - \top\llcorner_{\sharp \mathrm{W}_d}\right| + \left|\top\llcorner_{\equiv \mathrm{W}_\kappa} - \top\llcorner_{\equiv \mathrm{W}_d}\right|\right)$$

Step 5: Based on these distance values, we find the score value with the help of the averaging techniques. We take the average of both distance values in the context of the averaging and geometric operators. The mathematical representation of the score value is represented as:

$$\mathcal{S}_\varrho = \frac{1}{n}\sum_{\gamma=1}^{n} d(\mathrm{W}_\mathcal{s}, \mathrm{W}_d)$$

Step 6: After the calculations of the score values, we give the grade of each alternative based on its performance in the whole process of this method. We select the optimal solution based on the score values. High score values represent the best choice, and low score values represent the worst alternative. Figure 1 is the graphical representation of the MABAC model steps.

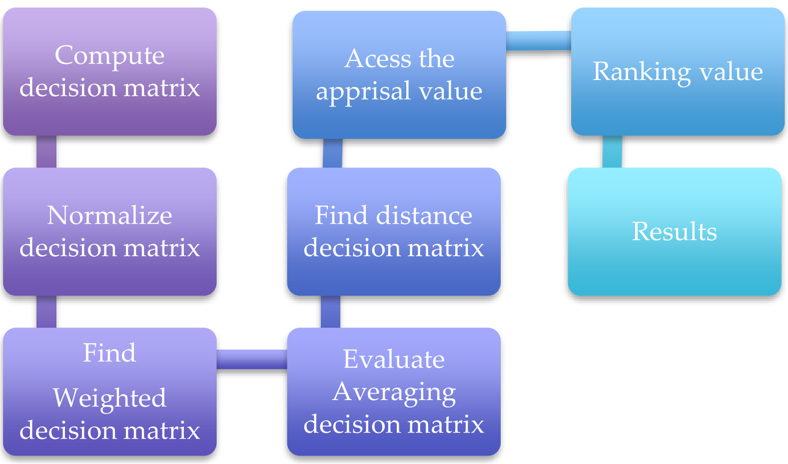

Figure 1. Graphical Representation of MABAC Model.

4. Examination of the Fault Detection Method in Wheel Bearing

In a rotating machine, wheel bearings play the most crucial role. Due to this, the machine parts rotate in a very good and smooth way. When this part is affected for any reason, then the whole machine's performance is affected. The identification of the faulty bearing is that it produced more noise and vibration. This is the main cause of the energy loss. If this fault is not detected early, then it may be the cause of the machine failure. The simple meaning of the fault detection is to identify the problems in the bearing before serious damage happens. The main benefits of early detection are that we can save costs and time. Many factories and industries use fault detection machines, which are very helpful to avoid unexpected shutdowns. Many industries use this early fault detection machine, such as the power plant and transportation industries. The simple method to detect this fault is visual inspection. Visual inspection could not identify the internal bearing faults. For this reason, many industries and factories do not depend on visual inspection; they prefer the advanced machinery to detect the fault in a very good and accurate way. Many individuals detect this fault due to the vibration pattern change, but the industries prefer the acoustic signal analysis techniques. They detect the sound signal generated by the faulty bearings. Overall, the fault detection techniques protect the vehicle, machine, and workers. The identification techniques increase the life of the machine and play the most fundamental role in reducing breakdowns. These techniques improve the reliability and productivity of the industries. Now we explain some alternatives to the fault detection techniques.

1) Vibration Analysis Techniques
   This is a general technique of fault detection of the wheel bearing; many people and experts use this technique. When the machine is rotating, it produces the vibration signals. The healthy and weak bearing produce the smooth and rough vibration patterns, respectively. Faulty bearings create the irregular vibration pattern, while the stable bearing produces the best and most regular vibrations. We can identify the bearing damage with the help of these vibrations. Many industries and manufacturing companies install the sensor near the bearing and collect all the data. With the help of the frequency analysis, we can identify the fault locations. These sensors detect the inner as well as the outer ball defects. This technique is very helpful for the early detection of faults. This technique is highly reliable and widely used in the big manufacturing industries.
2) Acoustic Emission Analysis
   In this way, we use the sound waves that come from the wheel bearings. When any faults exist, then these bearings produce the sound, and often these sound waves are not audible to human ears. In this method, very special sensors are installed, which have the capacity to capture the acoustic signal. This machine can detect very low faults at the early stages and also work very effectively when the machine is running at a very low speed. This machine uses the noise filtering technique, which plays a fundamental role in

improving the accuracy. This technique is very useful for a high-accuracy and precision monitoring system.

3) Temperature Monitoring Method
We can also detect the fault with the help of the surface temperature. Best and healthy bearing operates in the very best and normal temperature range, but a faulty bearing always produces more heat due to the friction. This type of machine only captures the irregular heat rise, and it gives a signal when serious damage occurs. This technique is very easy and simple for the implementation of any type of wheel bearing. This machine is very economical and requires the very-low low-cost sensors. These techniques are always combined with the vibration analysis techniques. This technique is very helpful to prevent overheating and sudden failure due to wheel bearing defects.

4) Current Signal Analysis
This is also the best and most modern technique for the early detection of wheel bearing, especially in big vehicles and cars. When any bearing has a fault, then this machine changes the current pattern and affects the motor load. This fault may cause small changes in the current in the form of the waveforms. We can see these changes due to the signal processing. Due to this one, we do not need any additional sensors. This method is called non-invasive for the fault detection of the wheel bearings. This method is suitable for all electrical machines. We can maintain this machine in a very easy way due to its small size. We can detect the fault in the wheel bearing in a very good and accurate way, especially in an inaccessible bearing system. This technique is very helpful for the motor industry.

5) Artificial Intelligence (AI)-Based Techniques
Using the AI, we can detect the early fault in the wheel bearing because it uses a smart algorithm for the identification of the fault. We can train the machine on the big data using different algorithms and learn the pattern. Deep learning techniques always deal with uncertain and complex data. With the help of machine learning techniques, we can automatically classify the different faulty bearing conditions. Due to this, AI techniques always reduce human involvement, and machines give an automatic result of the fault in the different machinery. AI has the capacity to capture multiple faults at the same time. AI plays the most important role in improving the accuracy and reliability of the fault detection techniques. Overall, this method represents the future of intelligent and best maintenance systems for wheel bearing systems. Table 1 is a representation of the advantages of the fault detection method of the wheel bearing.

Table 1. The representation of properties of the attributes with an example.

	Attributes	**Advantages**	**Practical example**
1	Vibration Analysis	This one contains high detection accuracy and gives reliable results.	We can use it in wind turbines and also use it in industrial machinery.

2	Acoustic Signal Analysis	Provides high-frequency fault detection and is best for precision monitoring.	This alternative is used in the quality control units.
3	Temperature Monitoring	We can use it in a very simple and easy way.	We can use this special alternative in the motors and power plants system.
4	Current Signal Analysis	We can install it in a very easy way and get a cost-effective solution.	This one is used for the automated production lines.
5	AI-Based Fault Detection	We get high accuracy with the real-time monitoring.	AI-based systems are used in smart factories and intelligent systems.

Now we discuss the attributes of the above alternatives. The Vibration analysis provides high detection accuracy and gives early fault bearing detection. These methods also give different operating speeds with strong reliability and high accuracy. Acoustic Emission analysis gives a high-frequency signal and early micro-fault detection with the best accuracy. These well-known techniques also improved the accuracy through noise filtering. Temperature Monitoring techniques are a very easy and cost-effective technique for fault detection. These techniques always track the temperature and detect overheating with optimal accuracy. Current signal techniques give cost efficiency and have the ability to detect the load variations. AI is the modern system that captures automatic fault classification and real-time monitoring capabilities. Now we discuss all these steps in the form of a mathematical representation as follows:

Step 1: First, we arrange all the alternatives and attributes of the fault detection techniques in Table 2. This matrix is said to be the decision matrix, and it is the initial step of the MABAC method. The data in Table 2 is benefit type, so do not need to normalize the decision matrix.

Table 2. Representation of a decision matrix.

	A1	**A2**	**A3**	**A4**	**A5**
$y_{\vert}$	$(\vert_{1}, \vert_{-6})$	$(\vert_{2}, \vert_{-6})$	$(\vert_{2}, \vert_{-4})$	$(\vert_{2}, \vert_{-5})$	$(\vert_{2}, \vert_{-1})$
y_2	$(\vert_{6}, \vert_{-7})$	$(\vert_{4}, \vert_{-5})$	$(\vert_{3}, \vert_{-3})$	$(\vert_{3}, \vert_{-4})$	$(\vert_{4}, \vert_{-2})$
y_3	$(\vert_{5}, \vert_{-2})$	$(\vert_{3}, \vert_{-2})$	$(\vert_{2}, \vert_{-2})$	$(\vert_{2}, \vert_{-2})$	$(\vert_{3}, \vert_{-2})$
y_4	$(\vert_{3}, \vert_{-2})$	$(\vert_{2}, \vert_{-2})$	$(\vert_{1}, \vert_{-2})$	$(\vert_{1}, \vert_{-2})$	$(\vert_{2}, \vert_{-2})$
y_5	$(\vert_{2}, \vert_{-4})$	$\vert_{3}, \vert_{-7}$	$(\vert_{6}, \vert_{-7})$	$(\vert_{2}, \vert_{-7})$	$(\vert_{1}, \vert_{-7})$

Step 2: With the help of the power and scaler multiplication techniques, we convert all the data into a fixed scale. For this, we apply these two normalization techniques to the decision matrix data and find the best and most accurate normalized values, see Table 3.

Table 3. Representation of a weighted decision matrix.

	A1	**A2**	**A3**	**A4**	**A5**
y_1	(| 1.875, | −4.5)	(| 3.5, | −4.5)	(| 3.5, | −2)	(| 3.5, | −3.125	(| 3.5, | −0.125)
y_2	(| 7.5, | −6.125)	(| 6, | −3.125)	(| 4.875, | −1.125)	(| 4.875, | −2)	(| 6, | −0.5)
y_3	(| 6.875, | −0.5)	(| 4.875, | −0.5)	(| 3.5, | −0.5)	(| 3.5, | −0.5)	(| 4.875, | −0.5)
y_4	(| 4.875, 𝔗 −0.5)	(| 3.5, | −0.5)	(| 1.875, | −0.5)	(| 1.875, | −0.5	(| 3.5, | −0.5)
y_5	(| 3.5, | −6.125)	(| 4.875, | −6.125)	(| 7.5, | −6.125)	(| 3.5, | −6.125	(| 1.875, | −6.125)

Step 3: Now apply the LBFPWA and LBFPWG operators on the normalized data and get the flexible and consistent aggregated values, see Table 4.

Table 4. LBFP aggregated operator.

	LBFPWAA	**LBFPWGA**
y_1	(| 5.7834, | −2.1351)	(| 4.3634, | −4.3195)
y_2	(| 4.692446, | −1.85336)	(| 4.449645, | −3.52103)
y_3	(| 4.830067, | −1.17354)	(| 3.715082, | −2.32031)
y_4	(| 3.559131, | −1.55726)	(| 3.3116, | −2.96064)
y_5	(| 4.291183, | −0.56119)	(| 3.792223, | −1.93521)

Step 4: This is a very important step in this model. In this step, we find the distance between step 2 and step 3. In this way, we calculate how much each alternative is from the ideal boundary, see Table 5.

Table 5. Representation of distance measure.

	A1		**A2**		**A3**		**A4**		**A5**	
y_1	0.392086	0.16681	0.239943	0.120539	0.134783	0.033462	0.101679	0.022047	0.076711	0.131402
y_2	0.35665	0.308877	0.161199	0.121649	0.005842	0.147202	0.109913	0.157752	0.110625	0.227686
y_3	0.170414	0.39569	0.095995	0.215399	0.125225	0.127212	0.0697750	0.165564	0.040313	0.157374
y_4	0.158975	0.27069	0.159113	0.248167	0.226788	0.228775	0.171337	0.243578	0.053273	0.107694
y_5	0.392086	0.16681	0.278387	0.189333	0.476337	0.47435	0.289179	0.209547	0.49875	0.381689

Step 5: Based on these distance values, we find the score value with the help of the averaging techniques. We take the average of both distance values in the context of the averaging and geometric operators, see Table 6, and also listed in Figure 2.

Table 6. Score values.

	Score values	
A1	0.294042	0.261775
A2	0.186927	0.179017
A3	0.193795	0.2022
A4	0.148377	0.159698
A5	0.155934	0.201223

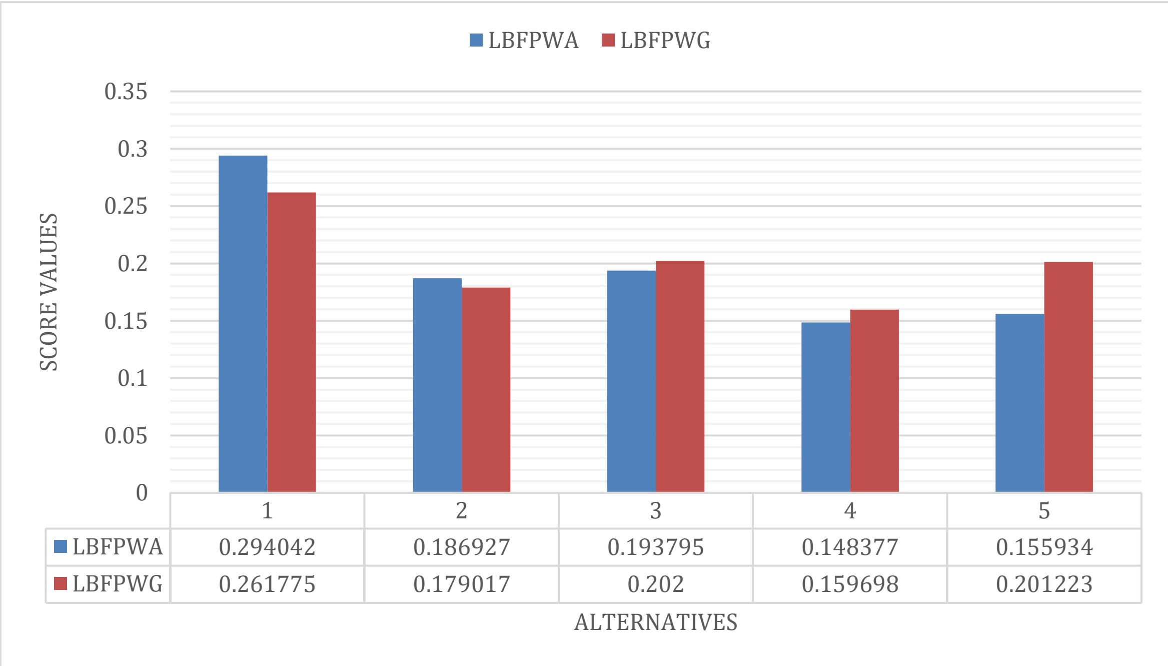

Figure 2. Graphical representation of the data in Table 6.

Step 6: After the calculations of the score values, we give the grade of each alternative based on its performance in the whole process of this method. We select the optimal solution based on the score values. High score values represent the best choice, and low score values represent the worst alternative, see Table 7.

Table 7. Ranking values information.

Methods	Ranking values
$\boldsymbol{MABAC-\mathbf{LBFSPWA}}$	$A1 > A3 > A2 > A5 > A4$
$\mathbf{MABAC-LBFSPWG}$	$A1 > A3 > A5 > A2 > A4$

The most valid decision is $A1$, and the worst one is $A4$ according to the technique of the MABAC model based on power aggregation operators. Further, we check the validity of the derived theory based on the decision-making procedure without the MABAC model. Therefore, the aggregated values are listed in Table 8.

Table 8. Without the MABAC Aggregation operator.

	LBFSPWA	LBCFSPWG
A1	$(\|3.789049, \|-4.13291)$	$(\|2.792669, \|-5.66309)$
A2	$(\|2.85603, \|-3.85057)$	$(\|2.701048, \|-5.07006)$
A3	$(\|2.964182, \|-3.06403)$	$(\|2.238706, \|-3.97555)$
A4	$(\|2.039551, \|-3.52961)$	$((\|1.89509, \|-4.59286))$
A4	$(\|2.552933, \|-2.11885)$	$(\|2.246665, \|-3.35371)$

Thus, using the technique score function, the score values are listed in Table 9 and also listed in Figure 3.

Table 9. Score value for data for Table 7.

	LBFSPWAA	LBFSPWGA
A1	0.478509	0.0400
A2	0.437841	0.351937
A3	0.493759	0.391447
A4	0.406872	0.331389
A5	0.52713	0.43081

Based on the data in Table 9, the ranking values are listed in Table 10.

Table 10. Ranking information for Table 8.

Methods	Ranking values
LBFSPWA	$A5 > A3 > A1 > A2 > A4$
LBFSPWG	$A5 > A3 > A2 > A4 > A1$

The best decision is $A5$, and the worst one is different because of ambiguity and problems. Further, we deliberated the validity and effectiveness of the invented theory based on the comparative analysis between proposed and existing ranking values to state the art of the derived theory.

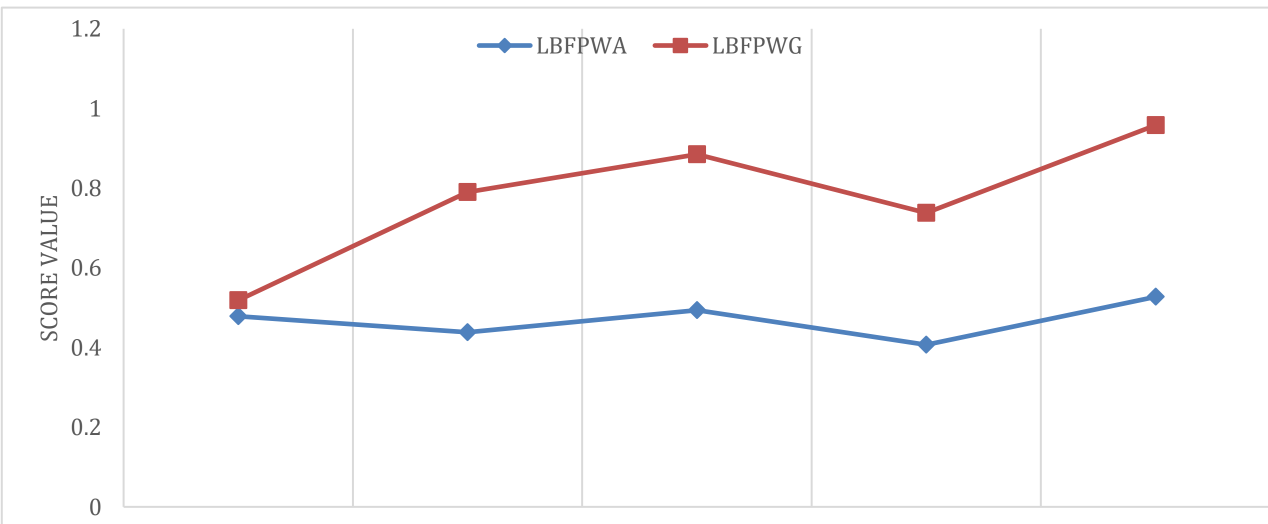

Figure 3. Graphical representation of score value from the data in Table 9.

5. Comparative Analysis

This section concentrates on the valuation of the comparative techniques or assessment by using the data in Table 2. The comparative assessment is an essential model for the valuation of the supremacy and validity of the proposed theory. For this, we arranged some existing models based on fuzzy models and their extensions to compare the ranking values of the existing models with the ranking values of the proposed theory. Therefore, we considered the following: Gul [19] constructed a novel VIKOR approach under an extended bipolar fuzzy enviro≡ent for handling multi-criteria decision-making (MCDM) problems. Gul et al. [20] established an extended bipolar fuzzified approach under bipolar fuzzy preference relations and discussed its applications in a decision-making enviro≡ent. Dalkılıc and Demirtas [21] constructed a novel decision-making algorithm under an extended bipolar domain for medical diagnosis. Alkouri et al. [22] developed multi attribute decision making approach under an extended bipolar fuzzy domain to find an optimal nutrition program. Ahmad et al. [23] constructed a novel decision-making approach under a generalized bipolar fuzzy enviro≡ent for sustainable energy solutions. Akram and Akmal [24] extended the application of BPS to the graph structural enviro≡ent. Finally, using the data in Table 2, the comparative assessment is listed in Table 11.

Table 11. Comparative Analysis

Method	Score value	Ranking values
Gul [19]	No	No
Gul et al. [20]	No	No
Dalkılıc and Demirtas [21]	No	No
Ahmed et al. [23]	No	No
Akram and Akmal [24]	No	No
MABAC-LBFSPWAA	0.2940,0.1869,0.1937,0.1483,0.1559	$A1 > A3 > A2 > A5 > A4$
MABAC-LBFSPWGA	0.2617,0.1790,0.2022,0.1596,0.2012	$A1 > A3 > A5 > A2 > A4$
LBFSPWA	0.4785, 0.4378, 0.4937, 0.4068, 0.5271	$A5 > A3 > A1 > A2 > A4$
LBFSPWG	0.0400, 0.3519, 0.3914, 0.3313, 0.4308	$A5 > A3 > A2 > A4 > A1$

The most valid decision is $A1$, and the worst one is $A4$ according to the technique of the MABAC model based on power aggregation operators. But using just the decision-making model, our best decision is $A5$, and the worst one is different because of ambiguity and problems. The existing techniques are not able to evaluate the data in Table 2, because they all cover the special cases of the proposed theory.

6. Conclusions

Finally, we concluded that the proposed work is very effective and reliable because of the validity and integration of the existing models. The major valuation of the proposed manuscript is listed as:

1) This study constructs the notation of linguistic bipolar fuzzy sets and discusses its basic properties.
2) This study developed a generalized idea of power aggregation operators based on linguistic bipolar fuzzy data.
3) This study invented an advanced notation of the MABAC model under a linguistic bipolar fuzzy enviro≡ent.
4) This study integrated the advanced idea of power aggregation operators with the MABAC model and applied it to a real-world problem.
5) This study compared the results of the proposed model with some related current models to check its reliability and accuracy.

In this future, we will work on the valuation of the weighted aggregated sum product assessment model based on LBFSs and also discuss their application in green supply chain, hydrogen energy, road signal systems, neural networks, and decision-making techniques to evaluate the supremacy and validity of the invented theory.

Author Contributions

The following statements should be used: "Conceptualization, M.A.; Z.A.; H.Z.; Y.T.C.; and A.K.; methodology, M.A.; Z.A.; H.Z.; Y.T.C.; and A.K.; software, M.A.; Z.A.; H.Z.; Y.T.C.; and A.K.; validation, M.A.; Z.A.; H.Z.; Y.T.C.; and A.K.; formal analysis, M.A.; Z.A.; H.Z.; Y.T.C.; and A.K.; investigation, M.A. and Z.A.; resources, M.A. and Z.A.; data curation, M.A. and Z.A.; writing—original draft preparation, M.A.; writing—review and editing, Z.A.; visualization, M.A.; Z.A.; H.Z.; Y.T.C.; and A.K.; supervision, Z.A.; project administration, Z.A.; funding acquisition, Z.A.

Funding

This work was supported in part by the National Science and Technology Council, Taiwan, under Grant NSTC 114-2410-H-224-001 and internal number 114-1011.

Data Availability Statement

Not applicable.

Conflicts of Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgement

This work was supported in part by the National Science and Technology Council, Taiwan, under Grant NSTC 114-2410-H-224-001 and internal number 114-1011.

References

[1] Cao, H., Fan, F., Zhou, K., & He, Z. (2016). Diagnosis of wheel bearing faults in trains using empirical wavelet transform. *Measurement*, *82*, 439-449. https://doi.org/10.1016/j.measurement.2016.01.023

[2] Shaikh, M. Z., Jatoi, S., Baro, E. N., Das, B., Hussain, S., & Chowdhry, B. S. (2025). FaultSeg: A Dataset for Train Wheel Defect Detection. *Scientific data*, *12*(1), 309. https://doi.org/10.1038/s41597-025-04557-0

[3] Nawrocki, W., Strisik, R., Kostrzewski, M., Woźniak, W., & Jachowicz, T. (2023). Utilization of Vibro-Acoustic Signals to Assess Wear in Spindle Bearings of Machining Centers. In-service diagnostics in the automotive industry. *Journal of Manufacturing Processes*, *92*, 165-178. https://doi.org/10.1016/j.jmapro.2023.02.036

[4] Yalçın, S. (2025). Detection of faults in tire steel belts utilizing magnetic sensors and various deep learning models. *Firat University Journal of Experimental and Computational Engineering*, *4*(1), 85-99. https://doi.org/10.62520/fujece.1527246

[5] Liu, C., Zhang, X., Wang, R., Guo, Q., & Li, J. (2024). Vibration-based detection of axle box bearings with inner and outer ring raceway defects. *Lubricants*, *12*(5), 142. **https://doi.org/10.3390/lubricants12050142**

[6] Pandiyan, M., & Babu, T. N. (2024). A systematic review of fault diagnosis on the rolling-element bearing. *Journal of Vibration Engineering & Technologies*, *12*(7), 8249-8283. https://doi.org/10.1007/s42417-024-01358-4

[7] Hossain, M. N., Rahman, M. M., & Ramasamy, D. (2024). Utilizing artificial intelligence vehicle fault diagnosis to transform automotive maintenance e: A Review. *CMES-Computer Modeling in Engineering & Sciences*, *141*(2). DOI: **10.32604/cmes.2024.056022**

[8] Shaikh, K., Hussain, I., & Chowdhry, B. S. (2023). On vehicle fault diagnosis driven by AI to revolutionize automotive maintenance. *Sensors*, *23*(14), 6248. **https://doi.org/10.3390/s23146248**

[9] Gruber, H., Fuchs, A., & Bader, M. (2024). Assessment of a condition monitoring algorithm for early detection of bearing faults. *Sensors*, *24*(7), 2138. **https://doi.org/10.3390/s24072138**

[10] Feng, H., Tao, Y., Feng, J., Zhang, Y., Xue, H., Wang, T., ... & Chen, P. (2025). Fault-Tolerant Collaborative Control of Four-Wheel-Drive Electric Vehicle for One or More In-Wheel Motors' Faults. *Sensors (Basel, Switzerland)*, *25*(5), 1540. doi: 10.3390/s25051540

[11] Zadeh, L. A. (|965). Fuzzy sets. *Information and control*, *8*(3), 338-353. https://doi.org/10.1016/S0019-9958(65)90241-X

[12] Başer, Z., & Uluçay, V. (2024). Properties and applications of Effective Q-fuzzy soft expert sets. *Uncertainty Discourse and Applications*, *1*(2), 195-209. https://doi.org/10.48313/uda.v1i2.41

[13] Kuntama, K., Krongchai, P., Prasertpong, R., Julatha, P., & Iampan, A. (2024). Apply fuzzy set on IUP- algebra. *Journal of Mathematics and Computer Science*, *34*(2), 128-143. doi: 10.22436/jmcs.034.02.03

[14] Lin, Y. C., Webster, A. J., Scruggs, C. E., Bixby, R. J., Cadol, D., Crossey, L. J., ... & Stone, M. C. (2025). Fuzzy SETS: Recognizing the multiple membership of elements within social-ecological-technological systems (SETS) theory. *Ecology and Society*, *30*(1). https://doi.org/10.5751/ES-15764-300122

[15] Vimala, J., Surya, A. N., Kausar, N., Pamucar, D., Simic, V., & Salman, M. A. (2025). An enhanced PROMETHEE method utilizing (p-q)-Rung linear Diophantine fuzzy sets for robot selection challenges. *Scientific Reports*, *15*(1), 69. https://doi.org/10.1038/s41598-024-81785-1

[16] Jadhava, V. S., Buktareb, S. U., & Jadhavc, O. S. (2024). Ranking octagonal numbers to address fuzzy job sequencing problems with a robust ranking technique. *Journal of Statistics, Optimization and Data Science*, *1*(2), 22-28. DOI: 10.1038/s41598-024-81785-2

[17] Castelló-Sirvent, F. (2022). Comparative analysis of publications on fuzzy set theory. *Mathematics*, *10*(8), 1322. **https://doi.org/10.3390/math10081322**

[18] Zhang, W. R. (|994, December). Bipolar fuzzy sets and relations: a computational framework for cognitive modeling and multiagent decision analysis. In *NAFIPS/IFIS/NASA'94. Proceedings of the first international joint conference of the North American Fuzzy Information Processing Society biannual conference. The Industrial Fuzzy Control and Intelligence* (pp. 305-309). IEEE. **DOI:** 10.1109/IJCF.1994.375115

[19] Gul, R. (2025). An extension of the VIKOR approach for MCDM using a bipolar fuzzy preference δ-covering based bipolar fuzzy rough set model. *Spectrum of Operational Research*, *2*(1), 72-91. https://orcid.org/0000-0002-6139-1872

[20] Gul, R., Shabir, M., & Naeem, M. (2023). A comprehensive study on (α, β)-bipolar fuzzified rough set model based on bipolar fuzzy preference relation and corresponding decision-making applications. *Computational and Applied Mathematics*, *42*(7), 310. https://doi.org/10.1007/s40314-023-02430-7

[21] Dalkılıc, O., & Demirtas, N. (2024). Worked in the medical field using a bipolar fuzzy set. *Turkish Journal of Mathematics and Computer Science*, *16*(2), 314-324. https://doi.org/10.47000/tjmcs.1254943

[22] Alkouri, A. U. M., Massa'deh, M. O., & Ali, M. (2020). On bipolar complex fuzzy sets and their application. *Journal of Intelligent & Fuzzy Systems*, *39*(|), 383-397. https://doi.org/10.3233/JIFS-191350

[23] Ahmad, Z., Ashraf, S., Khan, S., Tlija, M., Jana, C., & Pamucar, D. (2025). An enhanced decision model for sustainable energy solutions under bipolar hesitant fuzzy soft aggregation information. *AIMS MATHEMATICS*, *|0*(2), 4286-432|. doi: 10.3934/math.2025198

[24] Akram, M., & Akmal, R. (2016). Application of bipolar fuzzy sets in graph structures. *Applied Computational Intelligence and Soft Computing*, *2016*(1), 5859080. **https://doi.org/10.1155/2016/5859080**

[25] Zhang, W. R. (2006). YinYang bipolar fuzzy sets and fuzzy equilibrium relations: For clustering, optimization, and global regulation. *International Journal of Information Technology & Decision Making*, *5*(01), 19-46. **DOI:** 10.1109/FUZZY.1998.687599

[26] Akram, M., & Dudek, W. A. (2012). Regular bipolar fuzzy graphs. *Neural Computing and Applications*, *21*, 197-205. https://doi.org/10.1007/s00521-011-0772-6

[27] Dalkılıc, O., & Demirtas, N. (2024). Bipolar Fuzzy Soft Set Theory Applied to Medical Diagnosis. *Turkish Journal of Mathematics and Computer Science*, *16*(2), 314-324. https://doi.org/10.47000/tjmcs.1254943

[28] Zadeh, L. A. (1975). The concept of a linguistic variable and its application to approximate reasoning—I. *Information sciences*, *8*(3), 199-249. https://doi.org/10.1016/0020-0255(75)90036-5

[29] Gou, X., & Xu, Z. (2016). Novel basic operational laws for linguistic terms, hesitant fuzzy linguistic term sets, and probabilistic linguistic term sets. *Information Sciences*, *372*, 407-427. https://doi.org/10.1016/j.ins.2016.08.034

[30] Beg, I., & Rashid, T. (2013). TOPSIS for hesitant fuzzy linguistic term sets. *International Journal of Intelligent Systems*, *28*(12), 1162-1171. **https://doi.org/10.1002/int.21623**

[31] Rodríguez, R. M., Martínez, L., & Herrera, F. (2013). A group decision-making model dealing with comparative linguistic expressions based on hesitant fuzzy linguistic term sets. *Information Sciences*, *241*, 28-42. https://doi.org/10.1016/j.ins.2013.04.006

[32] Wang, H., Xu, Z., & Zeng, X. J. (2018). Hesitant fuzzy linguistic term sets for linguistic decision making: Current develo#ents, issues and challenges. *Information Fusion*, *43*, 1-12. https://doi.org/10.1016/j.inffus.2017.11.010

[33] Sidnyaev, N. I., Butenko, I. I., & Bolotova, E. E. (2020). Statistical and linguistic decision-making techniques based on fuzzy set theory. In *Advances in Intelligent Systems, Computer Science and Digital Economics* (pp. 165-174). Springer International Publishing. https://doi.org/10.1007/978-3-030-39216-1_16

[34] Pamučar, D., & Ćirović, G. (2015). The selection of transport and handling resources in logistics centers using Multi-Attributive Border Approximation Area Comparison (MABAC). *Expert systems with applications*, *42*(6), 3016-3028. https://doi.org/10.1016/j.eswa.2014.11.057

[35] He, M., Zhang, J., & Wu, X. (2020, August). Linguistic Value Bipolar Information Soft Rough Set and Its Application to Multi-criteria Group Decision Making. In *The International Conference on Natural Computation, Fuzzy Systems and Knowledge Discovery* (pp. 874-888). Cham: Springer International Publishing. https://doi.org/10.1007/978-3-030-70665-4_94

[36] Moslem, S. (2025). Evaluating commuters' travel mode choice using the Z-number extension of the Parsimonious Best Worst Method. *Applied Soft Computing*, *173*, 112918.

[37] Moslem, S. (2024). A novel parsimonious spherical fuzzy analytic hierarchy process for sustainable urban transport solutions. *Engineering Applications of Artificial Intelligence*, *128*, 107447.

[38] Badi, I., Bouraima, M. B., Yanjun, Q., & Qingping, W. (2025). Advancing Sustainable Logistics and Transport Systems in Free Trade Zones: A Multi-Criteria Decision-Making Approach for Strategic Sustainable Develo#ent. *International Journal of Sustainable Develo#ent Goals*, *1*, 45–55.

[39] Hussain, A., & Ali, M. (2025). A Critical Estimation of Ideological and Political Education for Sustainable Develo#ent Goals Using an Advanced Decision-Making Model Based on Intuitionistic Fuzzy Z-Numbers. *International Journal of Sustainable Develo#ent Goals*, *1*, 23–44.

[40] Nwokoro, C. O., & Ejegwa, P. A. (2025). A Decade of Picture Fuzzy Sets in Multi-Criteria Decision-Making: A Comprehensive Review of Trends, Gaps, and Future Directions. *Knowledge and Decision Systems With Applications*, *1*, 145–164.

[41] Zafer. H, Asif, M. (2025). Applications of N-structures to Fuzzy Graph Theory. Journal of Fuzzy Intelligence, 1(01), 31-40. https://mathfuzzyjournal.com/index.php/JFI/article/view/12

[42] Zhou, L., Abdullah, S., Zafar, H., Muhammad, S., Qadir, A., & Huang, H. (2025). Analysis of an artificial neural network based on pq-rung orthopair fuzzy linguistic Muirhead mean operators. Expert Systems with Applications, 276, 127157. https://doi.org/10.1016/j.eswa.2025.127157

[43] Jia, Z., Khan, S., Khan, N., Khan, B., & Asif, M. (2021). Faber Polynomial Coefficient Bounds for m-Fold Symmetric Analytic and Bi-univalent Functions Involving q-Calculus. Journal of Function Spaces, 2021(1), 5232247. https://doi.org/10.1155/2021/5232247